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Classical Differential Geometry I

Assignment 2, Part 1, 2019-20

Curves in space and in the plane

Curves in space

1. **Contact of curves:** We remind the reader that we say that two functions $f, g : \mathbf{R} \to \mathbf{R}$ **have contact of order** k at a point x_0 if at x_0 they have the same value, and the same derivatives of order up to k. A similar definition holds for space curves, on condition that they are expressed, locally, in the same affine coordinates.

   (a) What is the order of contact of the horizontal x axis with the graph of the function $f : \mathbf{R} \to \mathbf{R} : x \mapsto f(x) = x^4$? Find a parabola which has quadratic contact (i.e. contact of order 2) with the graph of f at the point $(x, y) = (1, 1)$ and give a suitable plot.

   (b) Consider the circle $(x-1)^2 + y^2 = 1$ and the 1-parameter family of ellipses
   $$\left(\frac{x-a}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$
   where $a = 3b$. Find the value of the semi-axis a such that the two curves have 2nd order contact at the point $(x, y) = (0, 0)$ and give a suitable plot.

   (c) Verify, from the definition of contact, that the tangent parabola $y = \kappa(s)x^2/2$ and the tangent circle $x^2 + (y - R(s))^2 = R(s)^2$ to a curve at a point $\boldsymbol{\rho}(s)$ where the curvature does not vanish indeed have contact of order 2. Here, the radius is the inverse of the curvature, $R(s) = 1/\kappa(s)$, and (x, y) are affine coordinates with respect to the affine basis $(\boldsymbol{\rho}(s), \mathbf{t}(s), \mathbf{b}(s))$ of the tangent plane at the point.

2. **The Viviani curve:** This is the curve obtained as the intersection of a sphere with a cylinder that is tangent to it from inside; in detail, it the section of the surfaces:
   $$x^2 + y^2 + z^2 = R^2 \text{ and } \left(x - \frac{R}{2}\right)^2 + y^2 = \left(\frac{R}{2}\right)^2.$$
   (Give a plot of these, preferably using some software.)

   Find a parametrization of this curve as follows: first, for the cylinder, use the parametrization $x(t) = (R/2) + (R/2)\cos(t)$, $y(t) = (R/2)\sin(t)$, $z = z$ (a simple translation of cylindrical coordinates.) Now substitute these into the equation of the sphere and find z by suitable use of trigonometric identities.

Show that we have a regular curve parametrization of a closed curve, except it is not 1:1 with its image, since there is a single point of self-intersection. Which point is this?

3. **Bézier curves:** In graphic design software (e.g. Adobe Illustrator, CorelDRAW etc) simple curves are used which have polynomial coordinates and can be stacked up to create more complicated curves. We use them as motivation for the following curves in space:

   A *linear Bézier curve* is the first-order curve $\mathbf{r}(t) = (1-t)\mathbf{v}_0 + t\mathbf{v}_1$, where $\mathbf{v}_0 \neq \mathbf{v}_1$ are two points in space. Obviously, $\mathbf{r}(0) = \mathbf{v}_0$, $\mathbf{r}(1) = \mathbf{v}_1$ and $\dot{\mathbf{r}}(0) = \mathbf{v}_1 - \mathbf{v}_0$.

   A *quadratic Bézier curve* uses three affinely independent points:

$$\mathbf{r}(t) = (1-t)\left[(1-t)\mathbf{v}_0 + t\mathbf{v}_1\right] + t\left[(1-t)\mathbf{v}_1 + t\mathbf{v}_2\right] =$$
$$= (1-t)^2\mathbf{v}_0 + 2t(1-t)\mathbf{v}_1 + t^2\mathbf{v}_2.$$

   Clearly, $\mathbf{r}(0) = \mathbf{v}_0$ and $\mathbf{r}(1) = \mathbf{v}_2$. Find the velocity vectors at $t = 0$ and $t = 1$ and show that the curve lies in the convex hull of the three points, in other words lies completely in the interior of the triangle they define, and is hence a plane curve. Explain how this curve could be continued from its end point with another quadratic curve, so that the velocity function is continuous.

4. We are given a regular curve parametrization in space

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ ax(t) + b \\ z(t) \end{bmatrix}, \ a \neq 0.$$

   Show that this curve has zero torsion and that it is therefore a plane curve, and find the plane containing it. Give a second, more direct proof that it is planar, by a suitable change of variables.

5. The *evolute* of a regular curve parametrization is defined as the curve obtained as the locus of the centres of the tangent circles $\boldsymbol{\rho}(s) + \frac{1}{\kappa(s)}\mathbf{n}(s)$. Show that for the helix $\mathbf{r}(t) = R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j} + vt\,\mathbf{k}$ ($R, \omega, v > 0$), the evoliute is again a helix.

   Find a condition so that this evolute lies on the same cylinder containing the original helix.

6. Show that the Frenet-Serre equations can be written in matrix form, by defining 3×3 matrices

$$\Phi(s) = [\mathbf{t}(s)|\mathbf{n}(s)|\mathbf{b}(s)], \ \Omega = \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\sigma(s) \\ 0 & \sigma(s) & 0 \end{bmatrix},$$

   as the system of ODEs

$$\frac{d}{ds}\Phi(s) = \Phi(s)\Omega.$$

   For a general skew-symmetric matrix

$$\Omega = \begin{bmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{bmatrix},$$

show how a vector $\boldsymbol{\omega}$ can be defined, so that for a vector $\mathbf{v} \in \mathbf{R}^3$,

$$\Omega \mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}.$$

Finally, show that a suitable vector function $\boldsymbol{\omega}(s)$ can be defined (the so-called Darboux vector)so that the Frenet-Serret equations can be written in the form:

$$\frac{d}{ds}\mathbf{t}(s) = \boldsymbol{\omega} \times \mathbf{t}, \quad \frac{d}{ds}\mathbf{n}(s) = \boldsymbol{\omega} \times \mathbf{n}, \quad \frac{d}{ds}\mathbf{b}(s) = \boldsymbol{\omega} \times \mathbf{b}$$

(the same $\boldsymbol{\omega}(s)$ for all three equations!)

7. (a) Show that the helix

$$\mathbf{r}(\theta) = \begin{bmatrix} v\theta \\ \cos\theta \\ \sin\theta \end{bmatrix},$$

where $v > 0$, lies on the cylinder $\{y^2 + z^2 = 1\} \subset \mathbf{R}^3$ and give its natural length parametrization. Find the Frenet frame, and the curvature function $\kappa(s)$. Hence, give the tangent plane and a parametrization of the tangent circle at an arbitrary point $\boldsymbol{\rho}(s_0)$ of the helix.

Show that the two parts of the helix $\{\boldsymbol{\rho}(s), s > s_0\}$ and $\{\boldsymbol{\rho}(s), s < s_0\}$ lie in opposite half-spaces whose boundary is the tangent plane at $\boldsymbol{\rho}(s_0)$.

(b) Find the value of the speed v so that the centre of each tangent circle lies on the cylinder containing the helix. Finally, compute the torsion from its definition from the Frenet-Serret equations

8. Show that is $h_1 : (a, b) \to \mathbf{R}$ and $h_2 : (a, b) \to \mathbf{R}$ are C^2 functions on the nonempty interval $(a, b) \subset \mathbf{R}$, then the parametrization

$$(a, b) \to \mathbf{R}^3 : x \mapsto \mathbf{r}(x) = x\,\mathbf{i} + h_1(x)\,\mathbf{j} + h_2(x)\,\mathbf{k}$$

is regular. Explain why the plane $x = c$, for $c \in (a, b)$, intersects the curve in a unique point.

Show that, conversely, if every plane x =const. intersects a regular curve γ in at most one point and does so trasversely (non-tangentially), then we can define functions h_1, h_2 as above, such that the curve admiuts a parametrization with parameter an interval of the x axis.

9. In this problem, we shall analyze the question of whether a curve can be defined as the intersection of two surfaces. This more traditional viewpoint is in gact quite problematic, as we shall see.

Let us assume that we have two "equations" in $\mathbf{R}^3$:

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

We assume that the set of solutions is non-empty. Here f_1 and f_2 are smooth scalar functions.

Consider the Jacobian 2×3matrix of first-order derivatives (the derivative), at each point of this non-empty set of solutions:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}.$$

Show that if the determinant of a 2×2 square sub-matrix is non-zero at a point, say for the sake of argument the first two columns, then functions g_1, g_2 are defined locally on some interval, and such that the parametrization $z \mapsto (g_1(z), g_2(z), z)$ gives a curve which is locally this set of solutions. In other words, the intersection of the surfaces is locally a curve (why can we in fact refer to surfaces in this case?) Explain why this condition is equivalent to the linear independence of the gradient vectors of the two functions at the point. Finally, find an equation for the velocity field of the intersection curve, based on the above.

Curves in the plane

We consider *regular curve parametrizations* in the plane $\mathbf{R}^2$:

$$\boldsymbol{\gamma} : I \to \mathbf{R}^2, \quad t \mapsto \boldsymbol{\gamma}(t) = \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \text{ with } \frac{d\mathbf{r}}{dt} \neq \mathbf{0}\ \forall t.$$

The natural (length) parametrization $\boldsymbol{\rho}(s) = \mathbf{r}(t(s))$ exists, in principle, and hence $\frac{d\boldsymbol{\rho}}{ds} = \frac{d\mathbf{r}}{dt} / \left\| \frac{d\mathbf{r}}{dt} \right\|$. We always have the normal vector field defined by

$$\mathbf{n}(s) = J\frac{d\boldsymbol{\rho}}{ds}, \quad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is the rotation matrix by angle $\pi/2$ in the positive sense. In terms of this nar, the curvature is then defined by the relation $\frac{d^2\boldsymbol{\rho}}{ds^2} = \kappa(s)\mathbf{n}(s)$. Now it can be positive, zero, or negative.

We also have the relation between the rate of change of the angle of the tangent vector and the curvature:

$$\kappa(s) = \frac{d\theta}{ds}(s) \quad \text{hence:} \quad \Delta\theta[s_0, s_1] = \int_{s_0}^{s_1} \kappa(s)\,ds,$$

where we denote by $\Delta\theta[s_0, s_1]$ the change of angle on the interval.

We recall that the natural parametrization is usually impossible to find explicitly, though we know it exists, since we rarely have the inverse function $t(s)$, even when we manage to find the function $s(t)$.

10. Show, based on the above, that the curvature function can be computed directly from the original, non-natural parametrization, as follows:

$$\boxed{\kappa(t) = \frac{\frac{d^2\mathbf{r}}{dt^2} \cdot J\frac{d\mathbf{r}}{dt}}{\left\| \frac{d\mathbf{r}}{dt} \right\|^3}.}$$

11. Give formulae for the centre $\mathbf{C}(t)$ and the radius $R(t)$ of the tangent circle at a point $\mathbf{r}(t)$.

12. Let us apply the above for **spiral curves** in the plane.

   (a) It is common for a spiral to be given in polar form: we have, for example, the *Archimedes spiral*, $r = a\theta$ and the *exponential spiral* $r = ae^{b\theta}$, for parameters $a, b \neq 0$. Here, r is as expected the length $\mathbf{r}$, $r = \|\mathbf{r}\|$, but careful: the "angle" θ is meant to vary in $\mathbf{R}_+$ and not, as we are used, on an inetrval of length 2π.

   We shall work on the spiral of Archimedes, for $a = 1$. Show that the curve parametrization is given by:

$$\mathbf{r}(t) = \begin{bmatrix} t\cos t \\ t\sin t \end{bmatrix}, \ t \geq 0.$$

   Compute the length function $s(t)$ and give its graph. We assume, without proof, that it is not possible to find an explicit form for the inverse function $t(s)$.

   (b) Therefore, compute the curvature function $\kappa(t)$ from the given parametrization and show that it is monotone in $\{t \geq 0\}$.

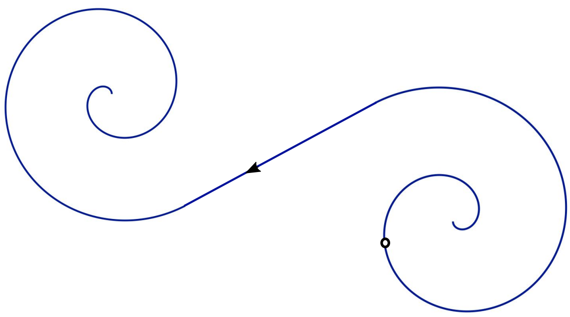
   (c) Find the centre and radius of the tangent circle to the spiral at $t = 4$ and give a parametrization for it. With the help of software, give a drawing of the spiral and the circle and notice that, for $t > 4$, the spiral lies in the exterior of the circle. Why do you think that is?

13. (a) Give a careful sketch of the regular C^1 curve in the plane whose curvature function with respect to the natural parametrization is:

$$\kappa(s) = \begin{cases} 1, & 0 < s < \pi \\ 3, & \pi < s < 4\pi/3, \\ -3, & 4\pi/3 < s < 5\pi/3 \\ 3, & 5\pi/3 < s < 2\pi \end{cases}$$

   Note that the curvature is undefined at the endpoints of these intervals. Is this curve closed?

   (b) Give a sketch of the curvature function of the C^2 planar curve shown. Also draw carefully the tangent circle at the marked point.



14. Suppose we have a regular smooth closed curve in the plane, in other word $\gamma : [a, b] \to \mathbf{R}^2$, with $\frac{d\mathbf{r}}{dt} \neq 0$ and $\mathbf{r}(a) = \mathbf{r}(b)$ and such that $\frac{d\mathbf{r}}{dt}(a) = \frac{d\mathbf{r}}{dt}(b)$ and which is nowehere zero, $\mathbf{r}(t) \neq \mathbf{0}$ for all t.

Consider times t_0 where the norm $\|\mathbf{r}(t_0)\|$ is minimal. Explain why such times must exist. Is t_0 unique? Give som,e suitable examples to explain. Then, show that the velocity vector $\frac{d\mathbf{r}}{dt}(t_0)$ is orthogonal to $\mathbf{r}(t_0)$ at each such time.

15. **The ellipse:** Give a regular parametrization $\mathbf{r}(t)$, $t \in (0, 2\pi)$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad a > b > 0$$

(use a variation of the usual polar coordinates.) Compute, from the formula you have derived before, the curvature function $\kappa(t)$ and identify the points of minimum and maximum curvature.

Derive the curve of the centres of the tangent planes and, using graphing software, give a drwaing of this curve. This curve is not regular, in general (why?) and is called the evolute of the ellipse.

EK, 28/12/2019