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RESEARCH ARTICLE

Topological Necessary Conditions for Control Dynamics

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We give a number of topological necessary conditions for the achievement of control dynamics more general than a single, asymptotically stable equilibrium point. In part, the results are based, as are the familiar results of Brockett and Coron, on simple considerations of Algebraic Topology. Since our setting is more global, though, we can go further with these tools and derive more elaborate necessary conditions. As a second simple technique, we also make use of the existence of Lyapunov functions to derive some conditions based on the implied homotopy between the control dynamics and the gradient flow of the Lyapunov function.

Keywords: Necessary conditions, topological index, Lyapunov functions.

1 Introduction

Suppose a smooth feedback control has been found so that the controlled dynamics have an asymptotically stable attractor at some point x in state space. Then *local Lyapunov functions* exist for the dynamics; these functions must all have a unique minimum at the point x, but are otherwise arbitrary. On a compact level set of any such Lyapunov function, the controlled dynamics point inwards, in other words in the direction of the negative of the gradient of the Lyapunov function. As maps to the unit sphere, the two vector fields thus have the same degree. But the degree of the negative gradient vector field is known —it is exactly $(-1)^n \neq 0$. Hence the map of the controlled dynamics, restricted to a level set, to the sphere must be **onto**. This means roughly that all control directions must be available near a point that is to be stabilized by control. In control theory, this is known as the Brockett condition, but such simple degree results were widely known before (Krasnosel'skii's name is mentioned in conjunction with that of Brockett.)

Such degree-theoretic arguments have been used for some time in topology and were eventually adopted by control theorists to derive a number of related *necessary conditions* for design of controlled dynamics using continuous feedback.

In this paper, we give an account of this theory that has two distinguishing features: first, there is really no reason to limit ourselves to *local results*: a **global** theory is straightforward to obtain. Secondly, we point out that such necessary conditions are in some fundamental sense of limited value; this is because they involve maps from manifolds to spheres of the same dimension. By the Hopf theory, homotopy equivalence classes of such maps are *completely classified by a single integer* (traditionally called the degree, but better interpreted in terms of homology groups.)

Several comments are in order: generalizations of the Brockett necessary condition have been obtained by Coron and others. These suffer to some extent by the problem mentioned above, but have helped clarify the fact that *surjectivity* is not enough: the vector field cannot 'twist' too

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much either (examples use the degree k maps $\dot{z} = z^k$ as part of a control system decomposition.) Moreover, extensions to the case of *dynamic feedback* have been derived.

Perhaps more importantly, many systems cannot be stabilized using continuous feedback, but can be easily stabilized with **discontinuous** feedback. The discontinuity is rather mild: it is usually limited to a 'thin' subset of state space. Recent work of Sontag, Clarke, Subbotin and others has led to a theory of discontinuous feedback controls and a methodology for obtaining nonsmooth Lyapunov functions. Now it is possible to interpret this theory in a *hamiltonian* context: The discontinuities correspond to jumps between locally nonsingularly projected lagrangian levels. This way of examining possibly discontinuous feedback controls is conceptually easier to understand and is in step with the philosophy of the book Kappos (2007) which is to give, as far as is possible, *geometric* accounts of *analytical* points.

We begin by giving an outline of the algebraic topological machinery needed for a discussion of necessary conditions. Here, we depart from the practice of delegating mathematical background to an appendix, because we believe that this theory is quite accessible and elegant.

A collection of *global necessary conditions* is then given, directly based on the topological results. Essentially, it is argued that if certain dynamics are achieved, then **index-theoretic** conditions can be deduced by counting the equilibrium points and their stability (Euler-type arguments) and *degree-theoretic* results are obtained by the Hopf theorems using the Gauss maps of the dynamics and the gradient vector field of Lyapunov functions.

Finally, let us point out that this paper by no means exhausts the theory of necessary conditions in control design. Much more crucial limitations on achievable control dynamics arise from the theory of **feedback invariant** objects, a theory that is developed in the book Kappos (2007).

2 Some Background and Methods from Algebraic Topology

A thumb-nail sketch of a number of concepts and methods from algebraic topology will now be given. There is no effort to be rigorous, but we do hope to explain enough about the computational methods so that a non-expert reader can use them in concrete situations.

Algebraic topology is based on a simple principle: attach algebraic objects to topological spaces that are invariants of the homotopy type of the space. Thus, more precisely we assign algebraic objects to homotopy equivalence classes of spaces and this assignment is 'functorial' in the sense that maps of spaces induce homomorphisms of the algebraic objects. This already gives useful tests: since homotopy equivalent spaces have isomorphic algebraic objects, two spaces are definitely not homotopy equivalent if their algebraic objects are not isomorphic. The bulk of algebraic topology consists of deriving finer and finer such objects so as to be able to better distinguish spaces and in making clever use of its basic constructions to aid the analysis of global aspects of other subjects (such as complex analysis, pdes, geometry etc.)

2.1 *Singular homology*

The easiest algebraic object we can attach to a space is the graded abelian group $H_*(X)$ called the **singular homology group** of X (with integer coefficients.) One can get quite far with only a vague understanding of what the singular homology measures and the reason is that powerful and effective methods for the computation of $H_*(X)$ exist. We outline the Mayer-Vietoris sequence and explain the concept of a long exact sequence of a pair and its relation to excision.

A graded abelian group $G = \oplus G_k$ is a direct sum of groups G_k, $k \in \mathbb{Z}_+$, such that the group addition is 'component-wise', in other words we add elements belonging to the same graded component together. The notation

$$g = \ldots + g^0 + g^1 + \ldots$$

for an element of G, where $g^k \in G_k$ is therefore unambiguous.

For a topological space, the kth homology group $H_k(X;\mathbb{Z})$ measures in some sense the *'holes'* of X that are like k spheres S^k (think of a boundary-less space, like a sphere, that does not actually bound anything itself in X.) The 0th group $H_0(X)$ is equal to $\mathbb{Z}$ if X is path connected. (Recall that a 0-sphere is the boundary of an interval, i.e. the union of two points.) There is a way of defining *reduced homology groups* $\tilde{H}_k(X)$ so that $\tilde{H}_0(X) = 0$ for a connected space and so that all higher dimensional groups coincide with the non-reduced ones.

Let us give some examples (we omit the zeroth homology group.) The singular homology of the circle S^1 is $H_1(S^1) \simeq \mathbb{Z}$ and zero for $k > 1$. Since $\pi_1(S^1) = \mathbb{Z}$ also, the homology group contains the same information as the fundamental group of the circle. Note the difference in interpretation, though: In the former case (for π_1), we are thinking of maps from the circle to itself, classifed by the number of net encirclements In the latter, we are thinking of a fixed circle —coinciding in this case with the whole space S^1— as the generator of a free abelian group; in this sense, we can write

$$H_1(S^1) = \mathbb{Z}[S^1] \simeq \mathbb{Z}.$$

For the sphere S^m of dimension $m > 1$, $H_m(S^m) \simeq \mathbb{Z}$ is the only nonzero homology group in positive dimension. Since we also have that the mth homotopy group of an m sphere is $\mathbb{Z}$, we have not yet obtained anything new, compared with homotopy theory. This is a little misleading: homotopy is both subtler than homology and far more difficult to compute: we do not, even today, have a complete list of the homotopy groups of spheres. Moreover, $\pi_{m+k}(S^m)$ may very well be nonzero for some $k > 0$, while $H_{m+k}(S^m) = 0$ always.

The 'coincidence' is really due to a nontrivial theorem, the **Hurewicz isomorphism** that states that homotopy and homology groups are isomorphic at the first level when one, and hence both, are nontrivial (the abelianization of the possibly nonabelian fundamental group is to be considered, if this happens at the first level.)

A quick check that homology theory does indeed give something new is to compute the homology of the torus T^2. We have that $H_2(T^2) \simeq \mathbb{Z}$ even though $\pi_2(T^2) = 0$! (it may be profitable to spend a minute or two pondering the difference.)

2.2 *The Mayer-Vietoris sequence*

Supose a space X is the union of two opne subsets, $X = A \cup B$, with A, B opne and $A \cap B \neq \emptyset$. Then there is a *long exact sequence* involving the homology groups of the three spaces

$$\ldots \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X) \to H_{k-1}(A \cap B) \to \ldots \tag{1}$$

This gives a surprisingly powerful tool for the computation of homology. Even without knowing what the maps at each stage are (for which we refer the reader to standard accounts such as Greenberg and Harper (1981)) the exactness allows the computation in concrete cases, such as that of the spheres. For this, decompose an m-sphere into two slightly overlapping hemispheres A and B so that their intersection $A \cap B$ is deformable to a sphere of dimension $m-1$. We can start an induction with dimension $m = 1$ and use the Mayer-Vietoris sequence to obtain

$$\ldots \to H_k(S^{m-1}) \to H_k(A) \oplus H_k(B) \to H_k(S^m) \to H_{k-1}(S^{m-1}) \to \ldots \tag{2}$$

yielding, for $k = m$, and since disks have no homology

$$\ldots \to 0 \to 0 \oplus 0 \to H_m(S^m) \to \mathbb{Z} \to 0 \to \ldots. \tag{3}$$

We conclude that $H_m(S^m) \simeq \mathbb{Z}$, since any exact sequence of the form

$$0 \to C \to D \to 0$$

implies that the middle map is one-to-one and onto, i.e. an isomorphism.

2.3 *Long exact sequence of a pair and excision*

A second very useful method for the computation of homology comes from considering pairs (X, A), where $A \subset X$ is a subspace. One gets the long exact sequence for the pair

$$\ldots \to H_k(A) \to H_k(X) \to H_k(X, A) \to H_{k-1}(A) \to \ldots \tag{4}$$

where the groups $H_k(X, A)$ are the **relative homology groups**. Without giving the exact definition, found in the standard texts, let us mention that in many important cases, these relative groups are isomorphic to the homology groups of the quotient space X/A (see chapter 3 for the definition.) The long exact sequence for a pair is thus extremely useful for the computation of the homological Conley index.

As an example, let us show that the quotient D^n/S^{n-1} of a closed ball by its bounding sphere has the homology of the n-sphere S^n. The long exact sequence of the pair (D^n, S^{n-1}) is

$$\ldots \to H_k(D^n) \to H_k(D^n, S^{n-1}) \to H_{k-1}(S^{n-1}) \to H_{k-1}(D^n) \to \ldots \tag{5}$$

and so, at $k = n$, we get

$$\ldots \to 0 \to H_k(D^n, S^{n-1}) \to \mathbb{Z} \to 0 \to \ldots \tag{6}$$

hence $H_k(D^n, S^{n-1}) \simeq H_k(D^n/S^{n-1}) \simeq \mathbb{Z}$. Similarly, one finds that, for $k \neq n$ (and nonzero), $H_k(D^n/S^{n-1}) = 0$.

2.4 *Maps and homomorphisms*

Given a continuous map $f : X \to Y$, there is an induced map in homology, which we shall denote by $H_*(f)$ or f_*

$$H_*(f) : H_*(X) \to H_*(Y)$$

and one checks that homology is a **covariant functor** from the category **Top** = (Top, C^0) of topological spaces and continuous maps to the category **Ab**= (Ab, Hom) of abelian groups and homomorphisms between them. Since $H_*(X)$ is graded, the above homomorphism is understood to mean that it consists of homomorphisms at each level of homology:

$$H_k(f) : H_k(X) \to H_k(Y)$$

for all k.

In fact, it would be more precise to say that the functor goes from the category **hTop** of homotopy equivalence classes of spaces and homotopic maps to the category **Ab**, since

Proposition 1. *If the maps f and g are homotopic, then the maps in homology coincide: $f_* = g_*$.*

and

Proposition 2. *If two spaces X and Y are homotopy equivalent and the map f has a homotopy inverse, then f_* is an isomorphism and the homology groups of X and Y, $H_*(X)$ and $H_*(Y)$ are isomorphic.*

Corollary 1. *If f is a homeomorphism of spaces, then f_* is an isomorphism.*

When the space X is a finite-dimensional manifold, the homology groups $H_k(X;\mathbb{Z})$ are finitely generated; thus, in this case, the basic structure theorem for finitely generated abelian groups is applicable.

Theorem 1. *Any finitely generated abelian group G decomposes uniquely as the direct sum*

$$G = F \oplus \tau$$

where the abelian group F is free and the group τ is a torsion subgroup.

In fact, one can describe the torsion group τ is more detail (see, for example, Lang (1971).)

The dimension of the free part of the homology group H_k is called the kth-**Betti number**, $b_k = \dim H_k(X;\mathbb{Z})$. The **Euler characteristic** $\chi(X)$ is the alternating sum of the Betti numbers

$$\chi(X) = \sum_k (-1)^k b_k.$$

3 Collections of topological necessary conditions

The definition of certain Gauss maps is helpful in the statement of our results. We shall assume that $M^n = \mathbb{R}^n$ or is an open subset of it.

Definition 1. (1) Suppose the vector field X is nowhere zero in M^n. Then the **Gauss map** $G_X : M^n \to S^{n-1}$ is defined by

$$x \mapsto \frac{X(x)}{|X(x)|}.$$

(2) Suppose that $N^{n-1} \subset M^n$ is a submanifold such that the restriction of the vector field X to N is nowhere zero. Then the **Gauss map** $G_{X|N} : N^{n-1} \to S^{n-1}$ is obtained by restricting the Gauss map G_X to N. Note that this is a map between two manifolds of the same dimension, one of which is a sphere.

(3) If the submanifold $N^{n-1} \subset M^n$ is orientable, we define the **Gauss map** $G_N : N^{n-1} \to S^{n-1}$ by mapping $x \in N$ to the unit normal vector to N at x (where an 'outward' direction is fixed by choosing an oriented basis on N and completing it to a basis of $\mathbb{R}^n$ consistent with an orientation of $\mathbb{R}^n$.) Note again that the Gauss map is a map from an $(n-1)$ dimensional space to the $(n-1)$-sphere.

3.1 *Index-Theoretic Necessary Conditions*

The *topological index* of equilibrium points leads to a number of necessary conditions for achieving dynamics with equilibrium points of given stability. These are *global* results and are rather classical; our only novelty is in trying to use as modern an algebraic topological framework as we can to express them.

If $e \in M^n$ is an isolated equilibrium point of the vector field X, take a ball neighborhood U of e (an open set homeomorphic to a ball) such that e is the *only* equilibrium of X in U and its boundary $N = \partial U$ is a closed submanifold homeomorphic to a sphere. Then the Gauss map

$G_{X|N}$ gives a map from the sphere S^{n-1} to itself

$$S^{n-1} \xrightarrow{h} N \xrightarrow{G_{X|N}} S^{n-1}$$

where h^{-1} is the homeomorphism from N to the sphere.

At the level of homology, we thus get a homomorphism $\psi = G_{X|N} \circ h$ from $H_{n-1}(S^{n-1})$ to itself. Since this group is isomorphic to $\mathbb{Z}$, we get a homomorphism from $\mathbb{Z} to \mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, such maps are specified by the image of the generator, say $\alpha \in H_{n-1}(S^{n-1})$. If, say, $\psi(\alpha) = k\alpha$, then k is the **topological index** of the equilibrium e.[1] It does not depend on the precise U chosen.

The classical theorem of Hopf describes maps from the sphere to itself.

Theorem 2. ***Hopf's Classification Theorem:*** *Homotopy equivalence classes of maps from S^{n-1} to itself are in a one-to-one correspondence with the integers. For each integer k, the class of maps corresponding to it is called the class of maps of* **degree** *k.*

For a hyperbolic equilibrium point of *stability index* k, the *topological index* (or degree) is equal to $(-1)^{n-k}$. Degree k maps are easily obtained from the degenerate equilibria at the origin of the system in complex form: $\dot{z} = z^k$, for $k \neq 0$.

The Hopf classification of maps from the sphere to itself has a crucial generalization to maps of an arbitrary compact manifold of dimension $n-1$ to a sphere of dimension $n-1$ (see Whitehead, Whitehead (1978), p.244)

Theorem 3 (Hopf-Whitney)**.** *The homotopy equivalence classes of maps of an $(n-1)$-dimensional compact manifold N^{n-1} to the sphere S^{n-1} are in one-to-one correspondence with the elements of the cohomology group $H^{n-1}(N^{n-1}; \mathbb{Z})$.*

Corollary 2. *If N is orientable, then the homotopy equivalence classes of maps from N^{n-1} to S^{n-1} are in one-to-one correspondence with the integers; they are thus again classified by 'degree.'*

This is, of course, because, for any orientable manifold, $H^{n-1}(N^{n-1}; \mathbb{Z}) \simeq \mathbb{Z}$. If N is **not orientable**, then this group is $\mathbb{Z}_2$ and two maps are homotopic iff they have the same mod-2 degree.

The **global** version of the Hopf index classification result is the following theorem of Poincaré-Hopf

Theorem 4 (Poincaré-Hopf)**.** *(1) Suppose $W^n \subset \mathbb{R}^n$ is a compact subset with nonempty interior such that its boundary is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Suppose X is a vector field on $\mathbb{R}^n$ that is nowhere zero on the boundary ∂W and has a finite set of equilibrium points E. Then*

$$\deg G_{X|\partial W} = \sum_{e_i \in E \cap W} \operatorname{ind} e_i \tag{7}$$

(2) Suppose M^n is a compact manifold and X is a vector field on M^n with a finite number of isolated equilibria. If the boundary of M^n is not empty, we require the vector field to point inwards at all points. The we have

$$\sum_{e_i \in E} \operatorname{ind} e_i = (-1)^n \chi(M^n) \tag{8}$$

[1]Confusingly, we are about to give a theorem where the term 'degree' is used instead of 'toplogical index'; the two terms are equivalent. We shall try use the qualifier 'topological' to avoid confusion with other uses of the term index.

where $\chi(M^n)$ is the Euler characteristic of the manifold M^n and E is the set of equilibrium points.

In particular, the sum of the topological indices of the equilibria is a topological invariant of the manifold and thus is independent of the vector field chosen.

(3) *Suppose W^k is any submanifold of $\mathbb{R}^n$, with $0 \leq k \leq n-1$. Consider a tubular neighborhood $N_\epsilon(W^k)$ so that $\partial N_\epsilon(W^k)$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. If X is any vector field on $\mathbb{R}^n$ such that, on W^k, X has a finite number of nondegenerate equilibria, then*

$$\sum_{e_i \in E \cap W} = \deg G_{X|N_\epsilon(W^k)} \tag{9}$$

We have collected different versions of this important theorem to help the reader find the most convenient form for extracting topological information in applications. Milnor Milnor (1965) proves versions (2) and (3) and contains a nice discussion.

Remark. The index already contains considerable topological information for the purposes of extracting necessary conditions. For the case of an asymptotically **attracting equilibrium**, for example, the topological index is equal to $(-1)^n$, which means that the generator of $H_{n-1}(S^{n-1})$ is mapped to itself or its negative, depending on the parity of n. As a result, the Gauss map is an isomorphism in homology and we conclude that it must then be surjective and injective. The surjectivity is essentially the **Krasnosel'skii-Brockett** condition and the injectivity was derived by **Coron**. The form we have given is, however, considerably more general.

Remark. It must be emphasized that the index is *'blind'* to all other dynamical features except equilibria. Looking at the same point from the other side of the equalities in Theorem 4, the topological type of the Gauss map in the large (on the boundary of an enclosing set) affects the configuration of equilibria inside—and fixes the sum of their indices.

A few examples as simple illustrations of the statements of the theorem:

Remark. In $\mathbb{R}^n$, a ball with a vector field pointing inwards at the boundary must contain equilibria whose index sum is $(-1)^n$. If these are all hyperbolic, then the options are

- A single attracting equilibrium.
- Two attractors and a one-saddle.
- If n is even, a single repeller is not ruled out; notice that the two cases can be distinguished using the Conley index, since the exit set differs for the two cases.
- Any other configuration of equilibria with the same net index sum.

Remark. In $\mathbb{R}^3$, an embedded **torus** T^2 gives possible Gauss maps of arbitrary degree, since its top homology is equal to $\mathbb{Z}$. If, however, we know that there are no enclosed equilibria, as for example in the case where the torus isolates a limit cycle, then the degree must be zero, by part (1) of Theorem 4, independently of the stability type of the limit cycle.

This means that the Gauss map is homotopic to the constant map and hence *does not have to be onto* (it is not onto in general, for a small enough torus around the limit cycle). Thus, no necessary condition is derivable in this case, whether the limit cycle is stable or not.

Remark. On the torus T^2 we have, by part (2) of the Theorem, that any vector field must have total index sum equal to zero, since the Euler characteristic of the torus is zero. Thus, vector fields that everywhere nonzero are permissible topologically, as are vector fields with one attractor and one saddle, one repeller and a saddle, one attractor, one repeller and two saddles etc.

3.2 *Necessary conditions using the topological index*

It should be clear from the examples how to derive necessary conditions for achieving global dynamics from the index theorems.

Suppose given a Morse specification of gradient type, $\mathcal{M} = (E, h_0)$, with Morse-lyapunov functions $\mathcal{F}(\mathcal{M})$. In the state space manifold, any choice of an oriented hypersurface that avoids $|E|$ has a Gauss map degree fixed by the sum of the indices of the enclosed 'equilibria'. If this is non-zero, this implies that there must exist control sections such that the controlled dynamics give a Gauss map with the desired property. In particular, if the index sum is equal to plus or minus one, then the Gauss map is onto. Let us remark that the conditions obtained can iether be used *locally* to check, for example, local stabilizability by requiring the map to have degree $(-1)^n$ for an arbitrarily small neighborhood of the equilibrium, or *globally*, since the only relevant information is the position and stability of the desired equilibria and hence the index/degree results hold for any compact hypersurface avoiding $|E|$.

A more elegant algebraic topological way of checking *simultaneously* all necessary conditions is the following (this does not make it easier to check in concrete cases):

We do first the case of local asymptotic stabilizability.

Suppose a control section $U \in \Gamma(D)$ is found that locally stabilizes the origin 0 in some neighborhood B; it will be helpful to consider the set, for $\epsilon > 0$,

$$\Sigma_B = \{(x, v) \in D|_B \ ; \ X(x) + v = 0\}$$

and the sequence

$$B \setminus \{0\} \xrightarrow{\text{graph}\, U} B \times \mathbb{R}^m \setminus \Sigma_B \overset{\iota}{\hookrightarrow} T\mathbb{R}^n|_B \setminus \{0\} \xrightarrow{G} S\mathbb{R}^n|_B \xrightarrow{\pi} S^{n-1}$$

where $\text{graph}\, U(x) = (x, U(x))$, ι is the inclusion map, G is the Gauss map and π is the obvious projection in the trivial local sphere bundle.

Since 0 is an isolated equilibrium of $X+U$, $X+U \neq 0$ in $B \setminus \{0\}$ and the above is well-defined.

3.3 *A reinterpretation of Coron's condition*

With the tools we have at our disposal, it is now easy to give a more geometric interpretation of the necessary condition for local feedback stabilization given in Coron (1990): We start by noticing that, if B is a ball neighborhood of the equilibrium 0, $B \setminus \{0\}$ is homotopically equivalent to S^{n-1} (it actually retracts to the sphere). Thus the composed map defined by the above sequence, call it ϕ,

$$\phi : B \setminus \{0\} \to S^{n-1}$$

has a well-defined degree, since 0 is asymptotically stable for $X + U$ and this degree is equal to $(-1)^n$. This means that, at the level of, for example, homology (or homotopy), the generator, call it α, of $H_{n-1}(S^{n-1}) \simeq \mathbb{Z}$ is in the image of ϕ. In other words, if 0 is LAS, then there is some local section such that the degree of the above map is defined and the image of the corresponding homomorphism at the level of homology is the whole of $H_{n-1}(S^{n-1})$. This is essentially Coron's result: Consider the commutative diagram

$$\begin{array}{ccc} B \setminus \{0\} & \to & S^{n-1} \\ \downarrow & \nearrow & \\ D \setminus \Sigma_V & & \end{array} \qquad (10)$$

where the vertical map is inclusion and the map from $D_B \setminus \Sigma_B$ to S^{n-1} will be denoted also by $X+U$ and is given by the composition $(x,v) \mapsto X(x)+v \mapsto G(X(x)+v)$. We have that

$$\phi_*(H_{n-1}(B \setminus \{0\})) = H_{n-1}(S^{n-1}).$$

Theorem 5 (Coron, 1990)**.** *If the system (X,D) is locally asymptotically stabilizable, then*

$$(X+U)_*(H_{n-1}(D_B \setminus \Sigma_B)) = H_{n-1}(S^{n-1}).$$

3.4 *Generalizations*

The simple reasoning that led to Coron's result can be generalized to equilibrium points that are not attractors, but have a well-defined stability index.

Theorem 6. *Let 0 be an equilibrium of the state dynamics X of the control pair (X,D). If there is a continuous local feedback that yields dynamics $X+U$ with 0 an equilibrium of index k, $0 \le k \le n$, then*

$$(X+U)_*(H_{n-1}(D_B \setminus \Sigma_B)) = H_{n-1}(S^{n-1}).$$

Finally, necessary conditions applicable to an arbitrary compact, connected IIS $\mathcal{S}$, isolated by the set $B \subset M^n$ can be given. More explicitly, we assume that there is a local feedback $U: B \to D$ such that $X+U$ has an IIS $\mathcal{S}$, whose dynamical structure is known (for example, $\mathcal{S}$ as a set consists of a number of equilibria and limit cycles and their connecting orbits.) Notice that $X+U \neq 0$ in $B \setminus \mathcal{S}$.

Remark. Endow M^n with a Riemannian metric. There is a function h defined on $B\setminus\mathcal{S}$ such that its gradient vector field ∇h is topologically equivalent to $X+U$ and such that the Gauss maps of ∇h and X_U induce the same homomorphisms on homology, both $G_{\nabla h}$ and G_{X_U} mapping

$$H_*(V \setminus \mathcal{S}) \to H_*(SM^n|_{V\setminus\mathcal{S}}).$$

Now, as we did for the case of local stabilization, we have the map ϕ defined by the composite map below

$$D_{B\setminus\mathcal{S}} \setminus \Sigma \overset{X+v}{\to} TM^n \setminus \Sigma \overset{G}{\to} SM^n \tag{11}$$

which induces the map ϕ_* in homology

$$H_*(D_{B\setminus\mathcal{S}}) \to H_*(SM^n|_{B\setminus\mathcal{S}}).$$

We now have the result

Theorem 7. *If the control pair (X,D) can achieve dynamics with IIS $\mathcal{S}$ isolated by the set B, then the images of the maps ϕ_* and $G_{-\nabla h}$ in $H_*(SM^n|_{B\setminus\mathcal{S}})$ coincide.*

4 Homotopy equivalence and homotopic results

An elementary, but fundamental result forms the key to an alternative approach to the derivation of necessary conditions. It concerns the Gauss maps of a gradient vector field of a Lyapunov function for the dynamics X and the Gauss map of the dynamics on level sets of the Lyapunov function.

Theorem 8. *Let $\mathcal{V}^{n-1}$ be a compact regular level set of some Lyapunov function V for the dynamics X on $M^n \subset \mathbb{R}^n$. Then, the Gauss maps $G_{X|\mathcal{V}}$ and $G_{-\nabla V|\mathcal{V}}$ os the vector field X and of the gradient vector field of V with respect to any riemannian metric are homotopy equivalent.*

Proof Decompose the tangent space $TM^n|_{\mathcal{V}}$ into the tangent space of $\mathcal{V}$ and the span of the gradient vector field ∇V. If X_n is the projection of X to the span of ∇V,we have that X_n is nowhere zero on $\mathcal{V}$.

Consider the isotopy of vector field

$$Y_t(x) = (1-t)X_n(x) + tX(x),\ 0 \leq t \leq 1.$$

We have that $Y_0 = X_n$ and $Y_1 = X$.

Now notice that this gives an isotopy for the corresponding Gauss maps as well: this is because $Y_t(x) \neq 0$ on $\mathcal{V}$ and for all t. To see this, write Y_t as

$$Y_t = X_n + t(X - X_n)$$

and notice that the vector field $X - X_n$ is orthogonal to X_n, which is everywhere nonzero.

Define the Gauss maps parametrized by t

$$G_t : \mathcal{V}^{n-1} \to S^{n-1},\ x \mapsto \frac{Y_t(x)}{|Y_t(x)|}.$$

Since $Y_t(x)$ is everywhere nonzero, this is well defined and gives an isotopy between

$$G_0 = \frac{X_n}{|X_n|} = \frac{-\nabla V}{|-\nabla V|} = G_{-\nabla V}$$

and

$$G_1 = \frac{X}{|X|} = G_X.$$

For reference purposes, let us denote the set of homotopy equivalence classes of maps between two spaces Ω and Ω' by

$$[\Omega, \Omega']$$

according to the standard notation. Given a map $f : \Omega \to \Omega'$, we write $[f]$ for its equivalence classs. We thus have, in this notation, that

$$[G_X] = [G_{-\nabla V}], \text{in } [\mathcal{V}^{n-1}, S^{n-1}].$$

4.1 *Relations to the index*

Since the spaces involved are of the same dimension and the target space is a sphere, we have, by the Hopf theory, that these homotopy equivalence classes are classified by degree.

4.2 *Limit Cycles*

In the case of a limit cycle γ, we saw that the Gauss map always has degree zero. Additional necessary conditions are obtained by examining the Gauss map in more detail.

Theorem 9. *Suppose γ is a limit cycle for the dynamics X on $\mathbb{R}^n$. then*

(1) For any $\epsilon > 0$, there is a neighborhood $N_\delta(\gamma)$ such that

$$G_X(N_\delta(\gamma)) \subset N_\epsilon(G_X(\gamma)).$$

(2) The image $G_X(\gamma)$ is not contained in any hemisphere: in other words, for any hyperplane $\mathcal{P} \subset \mathbb{R}^n$, $G_X(\gamma) \cap \mathcal{P} \neq \emptyset$. Moreover, for generic $\mathcal{P}$, $|G_X(\gamma) \cap \mathcal{P}|$ is even (here the bars denote cadinality of a finite set.)

Proof The first part is proved by continuity and the long flow box (see Palis and de Melo (1982).)

The second part is by contradiction: suppose there exists a hyperplane $\mathcal{P}_a = \{v \in \mathbb{R}^n \ ; \ a(v) = 0\}$, for some $a \in (\mathbb{R}^n)^*$ and is such that $G_X(\gamma) \cap \mathcal{P}_a = \emptyset$. Since any hyperplane separates S^{n-1} into two parts, we must have that $a(G_X(x))$ is of uniform sign, say negative, for all $x \in \gamma$.

Choose a basis $b_1, \ldots, b_n$ of $\mathbb{R}^n$ such that a is the dual basis vector of b_1, i.e. $a(b_1) = 1$ and $a(b_i) = 0$ for all $i \neq 1$. Write $x_1, \ldots, x_n$ for the coordinates in this basis.

Claim. *The function $V(x) = \frac{1}{2}x_1^2$ is a Lyapunov function for X in some open neighhborhood of γ.*

This is shown by computing $\frac{dV}{dt}|_\gamma$. We have

$$\frac{dV}{dt} = (x_1, 0, \ldots, 0) \cdot \dot{\gamma}$$

and, since $G_X = \frac{X}{|X|}$, this is just $a(X) < 0$.

The claim now establishes a contradiction that proves the theorem, since $G_X(\gamma)$ is a closed curve. The last part also follows from this fact and an elementary transversality argument.

Theorem 9 says roughly that, even though the image of the Gauss map of a limt cycle is 'thin,' still it must curve sufficiently in the target sphere so as to intersect all possible hyperplanes.

As for the Lyapunov level sets near a limit cycle, we have

Theorem 10. *Suppose γ is a stable limit cycle for some controlled dynamics. then, on each level set of a Lyapunov function near γ, each direction (i.e. element of the unit sphere) appears at least twice, in other words, for each $v \in S^{n-1}$,*

$$|G^{-1}_{-\nabla V}(v)| \geq 2.$$

(The proof is a basic topolgical facts about tori and is omitted.) Thus, even though the Gauss map of the gradient vector field of Lyapunov functions is of degree zero on any level (as it should be by the Poincaré-Hopf theorem 4), it covers the unit sphere at least twice.

We see, therefore, that members of the *same* homotopy equivalence class of maps can have widely different Gauss images. The trick, as far as control is concerned, is to find a representative arising from a control section (see Kappos (2007).)

5 Summary

We have presented ways of deriving collections of necessary conditions for achieving dynamics of a given type and we also pointed out the limitations of such topological conditions (due to the simplicity of the Hopf theory of maps to a sphere.) The basic aim of any analysis is, of course, to arrive at *constructive* methodologies. In the treatment of this subject in Kappos (2007), we find that conditions that are both necessary and sufficient can be found for achieving dynamics in a certain class. In this light, the fundamental source of necessary conditions is the class of control-transverse sections and the resulting feedback-invariant dynamics.

References

Coron, J.M. (1990), "A necessary condition for feedback stabilization," *System and Control Letters*, 14, 227–232.

Greenberg, M., and Harper, J., *Algebraic topology: a first course*, Benjamin/Cummins, Reading, Mass. (1981).

Kappos, E., *Global Controlled Dynamics*, Draft Manuscript (2007).

Lang, S., *Algebra*, Addison-Wesley, Reading, Mass. (1971).

Milnor, J.W., *Topology from a Differentiable Viewpoint*, The University Press of Virginia, Charlottsville (1965).

Palis, J., and de Melo, W., *Geometric theory of dynamical systems: an introduction*, Springer-Verlag (1982).

Whitehead, G.W., *Elements of homotopy theory*, Vol. 61 of *Graduate Texts in Mathematics*, Springer-Verlag (1978).