Aristotle University of Thessaloniki
Faculty of Pure Sciences, School of Mathematics,
Department of Geometry

Classical Differential Geometry I

Assignment 1, 2019-20

Background knowledge

The third year course in Classical Differential Geometry comes after several core course in the first two years, and is to some extent based on them. This first assignment goes over some of the prerequisites from these subjects. More specifically, we are assuming a good background in Linear Algebra, Calculus and Analytic Geometry.

It is assumed that basic competence on the techniques of CDG will be developed by the student through simple computational exercises, some found in the standard texts, and others made up by the student. The exercises below go a bit further, aiming at a deeper and more geometrical understanding. At several points, you are asked to provide graphs or pictures as part of your answers, so make sure you have access to suitable graphing software.

Geometry will take place in Euclidean space $\mathbf{R}^n$, most often just in $\mathbf{R}^3$, the set of all triples of real numbers, and where the inner product will be the usual **scalar** product: if $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ are elements of $\mathbf{R}^3$, their inner product is

$$\mathbf{u} \cdot \mathbf{v} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Most of the time, we shall consider the *standard* basis of $\mathbf{R}^3$, consisiting of the triples $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ and we shall write a point $\mathbf{r} = (x, y, z)$ of space as a column vector:

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is *orthonormal* with respect to the above inner product.
Careful: if we select a different basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathbf{R}^3$ (not necessarily orthonormal), then the same point will of course have different coordinates

$$\mathbf{r} = u\,\mathbf{b}_1 + v\,\mathbf{b}_2 + w\,\mathbf{b}_3 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_{\mathcal{B}}.$$

1. Show that the following vectors

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

give a basis of $\mathbf{R}^4$. Are they of the same, or the opposite orientation as the standard basis of $\mathbf{R}^4$, (e_1, e_2, e_3, e_4)? Find the 4-dimensional volume that they define, and explain why it is an integer. State a suitable generalization of this result.

Finally, express the standard, scalar inner product in $\mathbf{R}^4$ with respect to this basis, by defining a suitable 4×4 matrix.

2. In Euclidean space $\mathbf{R}^3$ we are given the vectors

$$\mathbf{b}_1 = (1, 3, -1), \quad \mathbf{b}_2 = (0, 1, 1), \quad \mathbf{b}_3 = (1, 1, 2).$$

   (a) Show that they give a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathbf{R}^3$.

   (b) Find the coordinates of the vector $\mathbf{v} = (-3, 7, 1)$ with respect to this basis.

   (c) Apply the Gram-Schmidt procedure to the ordered set $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ to find an orthonormal basis $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and find the coordinates of $\mathbf{v}$ with respect to this $\mathcal{E}$.

   (d) Find the orthogonal projection of $\mathbf{b}_2$ onto the plane $\mathrm{span}(\mathbf{b}_1, \mathbf{b}_3)$.

3. **Moving frames:** Let $f : \mathcal{D} \to \mathcal{E}$ be a C^1 function, where $\mathcal{D}, \mathcal{E}$ are open, nonempty subsets of $\mathbf{R}^n$, and which is 1:1, onto, and with a C^1 inverse $f^{-1} : \mathcal{E} \to \mathcal{D}$. The derivative $Df(\mathbf{x})$ at each point of $\mathcal{D}$ is given by a $n \times n$ matrix of first-order partial derivatives. Since we have that $f^{-1}(f(\mathbf{x})) = \mathbf{x}$, differentiating, we have that the derivative matrix is *invertible*, with inverse $(DF(\mathbf{x}))^{-1} = Df^{-1}(\mathbf{y}(\mathbf{x}))$, where $\mathbf{y} = f(\mathbf{x})$. in one dimension, this gives the well-known relation between the derivative of a function and the derivative of its inveres, $\frac{df^{-1}}{dy}(y_0) = 1/\frac{df}{dx}(x_0)$ (where $y_0 = f(x_0)$).

   It is convenient to consider the Cartesian products $\mathcal{D} \times \mathbf{R}^n$ and $\mathcal{E} \times \mathbf{R}^n$, so that we can refer to vectors in each vector space $\{\mathbf{x}\} \times \mathbf{R}^n$ and $\{\mathbf{y}\} \times \mathbf{R}^n$ (traditionally regarded as translated copies of $\mathbf{R}^n$ to the points $\mathbf{x}$ and $\mathbf{y}$ respectively, for each choice of points $\mathbf{x}$ and $\mathbf{y}$.) Show that at each point of the set $\mathcal{E}$ a basis of $\mathbf{R}^n$ is defined by the n columns of the derivative matrix. Since we have a basis at each point, and this point moves, we say that we get a *moving basis* of $\mathbf{R}^n$.

The familiar polar coordinates in the plane are worth a second look, as they provide one of the simplest examples of the difference between local and global inverse functions. The key is in the definition of *angle*, a tricky but very useful concept in many areas of mathematics.

4. **Polar coordinates:** Consider the open half-plane

$$\mathcal{D} = \{(r, \theta) \in \mathbf{R}^2 : r > 0\}$$

   and define a mapping

$$\phi : \mathcal{D} \to \mathbf{R}^2 : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta)).$$

   (a) Show that the condition of the Inverese Function Theorem holds at each point of $\mathcal{D}$ and hence, for each $(r_0, \theta_0) \in \mathcal{D}$ there is a neighborhood $\mathcal{U}$ (e.g. an open disk), such that ϕ gives a 1:1, onto map to the image $\phi(\mathcal{U})$, and there is a local inverse function, in other words a local change of variables.

Give the complete image of ϕ, $\phi(\mathcal{D})$ (the set of all its values.) Also describe the image of each horizontal half-line $\{(r, \theta) \in \mathcal{D} : \theta = \theta_0\}$ and of each vertical line $\{(r, \theta) \in \mathcal{D} : r = r_0\}$, and draw some typical such sets. If $(x_0, y_0) \neq (0, 0)$, find its inverse image, $\phi^{-1}(x_0, y_0)$.

Therefore, explain why we do not have a change of variables in all of the half-plane $\mathcal{D}$ and give reasons why we have to constrain the angle θ to an interval of total width 2π (e.g. $0 < \theta < 2\pi$).

(b) Give the images of the standard basis of the Euclidean plane $\mathbf{R}^2$, $(1, 0), (0, 1)$ under the derivative map to get the moving frame, as in the previous exercise. Give some indicative pictures at selected points.

5. Give the matrix A of the linear map $p \mapsto p'$ which represents differentiation of polynomials of degree at most n with respect to the basis $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$. Check that A^2 represents taking the derivative twice, and show that $A^{n+1} = 0$ (we say A is *nilpotent*.)

6. Give at least two distinct proofs of the inequality of Cauchy-Schwartz and, using it, give a proof of the triangle inequality in $\mathbf{R}^n$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Do not forget to give conditions for equality to hold.

7. **The exterior product in R^3.**

Historically, the definition of the exterior product in space is motivated by the desire to measure the total flow of a vector field through a surface in space. This physical problem is usually presented in a Vector Analysis course.

(a) One of the standard definitions of the exterior product is:

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta \, \mathbf{n}.$$

Geometrically, it gives the area of the parallelogram in space, whose sides are the two vectors, but includes a binormal vector $\mathbf{n}$, whose presence is unmotivated (but see definition of flow below). Explain why every single one of the four terms on the right (i.e. $\|\mathbf{u}\|, \|\mathbf{v}\|, \sin\theta, \mathbf{n}$), presupposes the existence of the inner product in $\mathbf{R}^3$. We wonder at this point whether we can define the exterior product as a measure of area, *but without reference to any inner product!*

(b) We claim that indeed the value of the exterior product depends only on the (oriented) area, and not on the length of the individual vectors, or on the angle between them! This should be clear from the following properties:

$$(a) \quad (k\mathbf{u}) \times (\mathbf{v}/k) = \mathbf{u} \times \mathbf{v}, \quad \forall k \neq 0$$

$$(b) \quad (\mathbf{u} + \alpha\mathbf{v}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}.$$

Make sure you know how they are proved and what they mean geometrically.

In fact, find the set of all pairs of vectors which give exactly the same value for their exterior product as the original pair $\mathbf{u}$ and $\mathbf{v}$.

(c) A better way to proceed to a definition independent of the inner product is the following: we shall accept that the area of the square defined by two of the basis vectors of the standard basis is equal to one, and the orientations are such that the resulting triple maintains the orientation of the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. This means that we define the elementary exterior products

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \ \mathbf{j} \times \mathbf{k} = \mathbf{i}, \ \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

and extend the definition to the exterior product of two general vectors by imposing the geometrically obvious properties of bilinearity and skew-symmetry (the first means that the product is linear in each factor, and the second that reversing the order reverses the resulting vector of the exterior product, $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$.)

Show that this gives the well-known alternative definition of exterior product in terms of coordinates with respect to the standard basis. We thus claim that this definition is preferable to the first one we gave above, since it does not assume any inner product.

(d) Recollect the definition of the flow of a vector field $\mathbf{F}$ through an oriented surface Σ:

$$\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS,$$

where the orientation of the syrface is given by the unit normal field $\mathbf{n}$. Explain why it is reasonable for the flow through an infinitesimal surface element to be given by $\mathbf{F} \cdot \mathbf{n} \, dS$ and relate it to the above definitions of the external product.

8. **Linearization:** We are given the function $f(x, y) = (x^3 - xy^2, -y^3 + xy)$. Compute its derivative Df as a 2×2 matrix. Show that f is locally invertible near the point $(x_0, y_0) = (2, 2)$.

   Compare the exact value of the difference $f(2.2, 1.8) - f(2, 2)$ with the approximate one, provided by the linearization:

$$f(x, y) - f(x_0, y_0) \simeq Df(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix},$$

   where, of course, here $(x, y) = (2.2, 1.8)$ and $(x_0, y_0) = (2, 2)$.

It is important to understand that on each nontrivial vector space, there are infinitely many different ways of defining an inner product, besides the standard, scalar one. We write the general inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.

9. Give full proofs of the following results:

   (a) The eigenvalues of a real square matrix are all real numbers.

   (b) A symmetric real matrix Q is called **positive definite** if for each vector $\mathbf{v} \neq 0$, the value of the quadratic forn $q(\mathbf{v}) = \mathbf{v} \cdot Q\mathbf{v}$ is strictly positive.

   (c) The following are equivalent conditions: (i) all its eigenvalues are positive and (ii) all the principal minors are positive (this is Sylvester's criterion.)

Now show that for any such positive definite Q, an inner product is defined in $\mathbf{R}^n$ by:

$$< \mathbf{u}, \mathbf{v} >_Q = \mathbf{u} \cdot Q\mathbf{v},$$

where $\cdot$ is the familiar scalar product.

Finally, show that the matrix

$$Q = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

is positive definite and find all vectors $\mathbf{v}$ orthogonal to $\mathbf{u} = (1, 2, 1)$ with respect to this inner product.

10. (a) Show that the function of two vectors:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2x_1x_2 - x_1y_2 - x_2y_1 + 3y_1y_2 + 2y_1z_2 + 2z_1y_2 + 5z_1z_2$$

defines an inner product in $\mathbf{R}^3$ (where we took $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$.

(b) Give the norm function $\| \cdot \|$ which is defined from this inner product. Hence, find the norm of the vector $(3, 3, -2)$ and give the equation of the plane orthogonal to this vector. How can you describe the set of all vectors in $\mathbf{R}^3$ which have norm one with respect to this IP?

11. (a) Compute the detrminant of the matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

where $a, b, c \in \mathbf{R}$ and therefore show that if a, b, c are distinct real numbers, the three row vectors give a basis of $\mathbf{R}^3$.

(b) Give ten distinct vectors in $\mathbf{R}^3$ suhc that any selection of *three* out of the ten gives a basis if $\mathbf{R}^3$.

12. We are given the smooth function $f : \mathbf{R}^2 \to \mathbf{R}^2 : (x, y) \mapsto (z, w) = (x^2 - y^2, 2xy)$.

(a) Show that near every nonzero point $(x, y) \neq (0, 0)$ it gives a local diffeomorphism (or change of variables) and that the moving frame we obtain by mapping the standard basis under the derivative map is actually an orthogonal basis, of the same orientation as the original.

(b) Observe that every pair of opposite nonzero points (x, y) kai $(-x, -y)$ give the same value of the function and show that, therefore, by limiting the domain of definition of f to the open upper half plane $\{y > 0\}$ we have a *global* diffeomorphism with its image, in other words, we have a global change of coordinates. Can you find the analytic form of the inverse function?

(c) Give an adequate geometric explanation for why we cannot have a change of variables in all of the punctured plane $\mathbf{R}^2 - \{0\}$, despite the fact that the function is locally invertible (in other words, how can we have a function $f : \mathbf{R}^2 - \{0\} \to \mathbf{R}^2 - \{0\}$ which is onto and locally 1:1, but is not globally invertible.)

EK, 27/10/2019